

SEXTIC VARIETY AS GALOIS CLOSURE VARIETY OF SMOOTH CUBIC

HISAO YOSHIHARA

*Department of Mathematics, Faculty of Science, Niigata University,
Niigata 950-2181, Japan*

E-mail: yosihara@math.sc.niigata-u.ac.jp

ABSTRACT. Let V be a nonsingular projective algebraic variety of dimension n . Suppose there exists a very ample divisor D such that $D^n = 6$ and $\dim H^0(V, \mathcal{O}(D)) = n + 3$. Then, (V, D) defines a D_6 -Galois embedding if and only if it is a Galois closure variety of a smooth cubic in \mathbb{P}^{n+1} with respect to a suitable projection center such that the pull back of hyperplane of \mathbb{P}^n is linearly equivalent to D .

1. INTRODUCTION

The purpose of this article is to generalize the following assertion (cf. [13, Theorem 4.5]) to n -dimensional varieties.

Proposition 1.1. *Let C be a smooth sextic curve in \mathbb{P}^3 and assume the genus is four. If C has a Galois line, then the group G is isomorphic to the cyclic or dihedral group of order six. Moreover, G is isomorphic to the latter one if and only if C is obtained as the Galois closure curve of a smooth plane cubic E with respect to a point $P \in \mathbb{P}^2 \setminus E$, where P does not lie on the tangent line to E at any flex.*

Before going into the details, we recall the definition of Galois embeddings of algebraic varieties and the relevant results. In this article a variety, a surface and a curve will mean a nonsingular projective algebraic variety, surface and curve, respectively.

Let k be the ground field of our discussion, we assume it to be an algebraically closed field of characteristic zero. Let V be a variety of dimension n with a very ample divisor D ; we denote this by a pair (V, D) . Let $f = f_D : V \hookrightarrow \mathbb{P}^N$ be the embedding of V associated with the complete linear system $|D|$, where $N + 1 = \dim H^0(V, \mathcal{O}(D))$. Suppose W is a linear subvariety of \mathbb{P}^N satisfying $\dim W = N - n - 1$ and $W \cap f(V) = \emptyset$. Consider the projection π_W from W to \mathbb{P}^n , i. e., $\pi_W : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$. Restricting π_W onto $f(V)$, we get a surjective morphism $\pi = \pi_W \circ f : V \rightarrow \mathbb{P}^n$.

Let $K = k(V)$ and $K_0 = k(\mathbb{P}^n)$ be the function fields of V and \mathbb{P}^n respectively. The covering map π induces a finite extension of fields $\pi^* : K_0 \hookrightarrow K$ of degree $\deg f(V) = D^n$, which is the self-intersection number of D . We denote by K_W the Galois closure of this extension and by $G_W = \text{Gal}(K_W/K_0)$ the Galois group of K_W/K_0 . By [1] G_W is isomorphic to the monodromy group of the covering $\pi : V \rightarrow \mathbb{P}^n$. Let V_W be the K_W -normalization of V (cf. [3, Ch.2]). Note that V_W is determined uniquely by V and W .

Definition 1.2. In the above situation we call G_W and V_W the Galois group and the Galois closure variety at W respectively (cf. [14]). If the extension K/K_0 is Galois, then we call f and W a Galois embedding and a Galois subspace for the embedding respectively.

Definition 1.3. A variety V is said to have a Galois embedding if there exist a very ample divisor D satisfying that the embedding associated with $|D|$ has a Galois subspace. In this case the pair (V, D) is said to define a Galois embedding.

If W is a Galois subspace and T is a projective transformation of \mathbb{P}^N , then $T(W)$ is a Galois subspace of the embedding $T \cdot f$. Therefore the existence of Galois subspace does not depend on the choice of the basis giving the embedding.

Remark 1.4. If a variety V exists in a projective space, then by taking a linear subvariety, we can define a Galois subspace and Galois group similarly as above. Suppose V is not normally embedded and there exists a linear subvariety W such that the projection π_W induces a Galois extension of fields. Then, taking D as a hyperplane section of V in the embedding, we infer readily that (V, D) defines a Galois embedding with the same Galois group in the above sense.

By this remark, for the study of Galois subspaces, it is sufficient to consider the case where V is normally embedded.

We have studied Galois subspaces and Galois groups for hypersurfaces in [9], [10] and [11] and space curves in [13] and [15]. The method introduced in [14] is a generalization of the ones used in these studies.

Hereafter we use the following notation and convention:

- $\text{Aut}(V)$: the automorphism group of a variety V
- $\langle a_1, \dots, a_m \rangle$: the subgroup generated by a_1, \dots, a_m
- D_{2m} : the dihedral group of order $2m$
- $|G|$: the order of a group G
- \sim : the linear equivalence of divisors
- $\mathbf{1}_m$: the unit matrix of size m
- $X * Y$: the intersection cycle of cycles X and Y in a variety.
- $(X_0 : \dots : X_m)$: a set of homogeneous coordinates on \mathbb{P}^m
- $g(C)$: the genus of a smooth curve C
- For a mapping $\varphi : X \rightarrow Y$ and a subset $X' \subset X$, we often use the same φ to denote the restriction $\varphi|_{X'}$.

2. RESULTS ON GALOIS EMBEDDINGS

We state several properties concerning Galois embedding without the proofs, for the details, see [14]. By definition, if W is a Galois subspace, then each element σ of G_W is an automorphism of $K = K_W$ over K_0 . Therefore it induces a birational transformation of V over \mathbb{P}^n . This implies that G_W can be viewed as a subgroup of $\text{Bir}(V/\mathbb{P}^n)$, the group of birational transformations of V over \mathbb{P}^n . Further we can say the following:

Representation 1. *Each birational transformation belonging to G_W turns out to be regular on V , hence we have a faithful representation*

$$\alpha : G_W \hookrightarrow \text{Aut}(V). \quad (1)$$

Remark 2.1. Representation 1 is proved by using transcendental method in [14], however we can prove it algebraically by making use of the results [7, Ch. I, 5.3. Theorem 7] and [2, Ch. V, Theorem 5.2].

Therefore, if the order of $\text{Aut}(V)$ is smaller than the degree d , then (V, D) cannot define a Galois embedding. In particular, if $\text{Aut}(V)$ is trivial, then V has no Galois embedding. On the other hand, in case V has an infinitely many automorphisms, we have examples such that there exist infinitely many distinct Galois embeddings, see Example 4.1 in [14].

When (V, D) defines a Galois embedding, we identify $f(V)$ with V . Let H be a hyperplane of \mathbb{P}^N containing W and put $D' = V * H$. Since $D' \sim D$ and $\sigma^*(D') = D'$, for any $\sigma \in G_W$, we see σ induces an automorphism of $H^0(V, \mathcal{O}(D))$. This implies the following.

Representation 2. *We have a second faithful representation*

$$\beta : G_W \hookrightarrow PGL(N + 1, k). \quad (2)$$

In the case where W is a Galois subspace we identify $\sigma \in G_W$ with $\beta(\sigma) \in PGL(N + 1, k)$ hereafter. Since G_W is a finite subgroup of $\text{Aut}(V)$, we can consider the quotient V/G_W and let π_G be the quotient morphism, $\pi_G : V \rightarrow V/G_W$.

Proposition 2.2. *If (V, D) defines a Galois embedding with the Galois subspace W such that the projection is $\pi_W : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$, then there exists an isomorphism $g : V/G_W \rightarrow \mathbb{P}^n$ satisfying $g \cdot \pi_G = \pi$. Hence the projection π is a finite morphism and the fixed loci of G_W consist of only divisors.*

Therefore, π turns out to be a Galois covering in the sense of Namba [6].

Lemma 2.3. *Let (V, D) be the pair in Proposition 2.2. Suppose $\tau \in G$ has the representation*

$$\beta(\tau) = [1, \dots, 1, e_m], \quad (m \geq 2)$$

where e_m is an m -th root of unity. Let p be the projection from $(0 : \dots : 0 : 1) \in W$ to \mathbb{P}^{N-1} . Then, $V/\langle \tau \rangle$ is isomorphic to $p(V)$ if $p(V)$ is a normal variety.

We have a criterion that (V, D) defines a Galois embedding.

Theorem 2.4. *The pair (V, D) defines a Galois embedding if and only if the following conditions hold:*

- (1) *There exists a subgroup G of $\text{Aut}(V)$ satisfying that $|G| = D^n$.*
- (2) *There exists a G -invariant linear subspace \mathcal{L} of $H^0(V, \mathcal{O}(D))$ of dimension $n + 1$ such that, for any $\sigma \in G$, the restriction $\sigma^*|_{\mathcal{L}}$ is a multiple of the identity.*
- (3) *The linear system \mathcal{L} has no base points.*

It is easy to see that $\sigma \in G_W$ induces an automorphism of W , hence we obtain another representation of G_W as follows. Take a basis $\{f_0, f_1, \dots, f_N\}$ of $H^0(V, \mathcal{O}(D))$ satisfying that $\{f_0, f_1, \dots, f_n\}$ is a basis of \mathcal{L} in Theorem 2.4. Then we have the representation

$$\beta_1(\sigma) = \begin{pmatrix} \lambda_\sigma & & & \vdots & \\ & \ddots & & \vdots & * \\ & & \lambda_\sigma & \vdots & \\ \dots & \dots & \dots & \vdots & \dots \\ & \mathbf{0} & & \vdots & M_\sigma \end{pmatrix}. \quad (3)$$

Since the projective representation is completely reducible, we get another representation using a direct sum decomposition:

$$\beta_2(\sigma) = \lambda_\sigma \cdot \mathbf{1}_{n+1} \oplus M'_\sigma.$$

Thus we can define

$$\gamma(\sigma) = M'_\sigma \in PGL(N - n, k).$$

Therefore σ induces an automorphism on W given by M'_σ .

Representation 3. *We get a third representation*

$$\gamma : G_W \longrightarrow PGL(N - n, k). \quad (4)$$

Let G_1 and G_2 be the kernel and image of γ respectively.

Theorem 2.5. *We have an exact sequence of groups*

$$1 \longrightarrow G_1 \longrightarrow G \xrightarrow{\gamma} G_2 \longrightarrow 1,$$

where G_1 is a cyclic group.

Corollary 2.6. *If $N = n + 1$, i.e., $f(V)$ is a hypersurface, then G is a cyclic group.*

This assertion has been obtained in [11]. Moreover we have another representation.

Suppose that (V, D) defines a Galois embedding and let G be a Galois group at some Galois subspace W . Then, take a general hyperplane W_1 of \mathbb{P}^n and put $V_1 = \pi^*(W_1)$. The divisor V_1 has the following properties:

- (i) If $n \geq 2$, then V_1 is a smooth irreducible variety.
- (ii) $V_1 \sim D$.
- (iii) $\sigma^*(V_1) = V_1$ for any $\sigma \in G$.
- (iv) V_1/G is isomorphic to W_1 .

Put $D_1 = V_1 \cap H_1$, where H_1 is a general hyperplane of \mathbb{P}^N . Then (V_1, D_1) defines a Galois embedding with the Galois group G (cf. Remark 1.4). Iterating the above procedures, we get a sequence of pairs (V_i, D_i) such that

$$(V, D) \supset (V_1, D_1) \supset \dots \supset (V_{n-1}, D_{n-1}). \quad (5)$$

These pairs satisfy the following properties:

- (a) V_i is a smooth subvariety of V_{i-1} , which is a hyperplane section of V_{i-1} , where $D_i = V_{i+1}$, $V = V_0$ and $D = V_1$ ($1 \leq i \leq n-1$).
- (b) (V_i, D_i) defines a Galois embedding with the same Galois group G .

Definition 2.7. The above procedure to get the sequence (5) is called the Descending Procedure.

Letting C be the curve V_{n-1} , we get the next fourth representation.

Representation 4. We have a fourth faithful representation

$$\delta : G_W \hookrightarrow \text{Aut}(C), \quad (6)$$

where C is a curve in V given by $V \cap L$ such that L is a general linear subvariety of \mathbb{P}^N with dimension $N - n + 1$ containing W .

Since the Inverse Problem of Galois Theory over $k(x)$ is affirmative ([4]), we can prove the following.

Remark 2.8. Giving any finite group G , there exists a smooth curve and very ample divisor D such that (C, D) defines a Galois embedding with the Galois group G .

3. STATEMENT OF RESULTS

Let V be a variety of dimension n . We say that V has the property (\P_n) if

- (1) there exists a very ample divisor D with $D^n = 6$, and
- (2) $\dim H^0(V, \mathcal{O}(D)) = n + 3$.

An example of such a variety is a smooth $(2, 3)$ -complete intersection, where D is a hyperplane section. In particular, in case $n = 1$, V is a non-hyperelliptic curve of genus four and D is a canonical divisor. In case $n = 2$, V is a $K3$ surface such that there exists a very ample divisor D with $D^2 = 6$. However, the variety with the property (\P_n) is not necessarily the complete intersection, see Remark 3.10 below.

We will study the Galois embedding of V for the variety with the property (\P_n) . Clearly the Galois group is isomorphic to the cyclic group of order six or D_6 . In the latter case we say that (V, D) defines a D_6 -embedding or, more simply V has a D_6 -embedding.

Theorem 3.1. Assume V has the property (\P_n) . If V has a D_6 -embedding, then V is obtained as the Galois closure variety of a smooth cubic Δ in \mathbb{P}^{n+1} with respect to a suitable projection center.

Next we consider the converse assertion. Let Δ be a smooth cubic of dimension n in \mathbb{P}^{n+1} . Take a non-Galois point $P \in \mathbb{P}^{n+1} \setminus \Delta$. Note that, for a smooth hypersurface $X \subset \mathbb{P}^{n+1}$, the number of Gaois points is at most $n + 2$. The maximal number is attained if and only if X is projectively equivalent to the Fermat variety (cf. [11]).

Define the set Σ_P of lines as

$$\Sigma_P = \{ \ell \mid \ell \text{ is a line passing through } P \text{ such that } \ell * \Delta \text{ can be expressed as } 2P_1 + P_2, \text{ where } P_i \in \Delta \text{ (} i = 1, 2 \text{) and } P_1 \neq P_2 \}$$

The closure of the set $\bigcup_{\ell \in \Sigma_P} \ell$ is a cone, we denote it by $C_P(\Delta)$. Then we have the following.

Lemma 3.2. The cone $C_P(\Delta)$ is a hypersurface of degree six.

We can express as $\Delta * C_P(\Delta) = 2R_1 + R_2$, where R_1 and R_2 are different divisors on Δ .

Definition 3.3. We call P a good point if

- (1) R_2 is smooth and irreducible in case $n \geq 2$, or
- (2) R_2 consists of six points in case $n = 1$.

Proposition 3.4. *If P is a general point for Δ , then P is a good point.*

To some extent the converse assertion of Theorem 3.1 holds as follows.

Theorem 3.5. *If Δ_P is a Galois closure variety of a smooth cubic $\Delta \subset \mathbb{P}^{n+1}$, where the projection center P is a good point, then Δ_P is a smooth $(2, 3)$ -complete intersection in \mathbb{P}^{n+2} with D_6 -embedding.*

Remark 3.6. In the assertion of Theorem 3.5, the construction of the Galois closure is closely related to the one in [8, Tokunaga]. In case $n = 2$, the Galois closure surface is a $K3$ surface.

Applying the Descending Procedure to the variety of Theorem 3.5, we get the following.

Proposition 3.7. *If a variety V is a smooth $(2, 3)$ -complete intersection and has a D_6 -embedding, then there exists the following sequence of varieties V_i , where V_i has the same properties as V does, i.e.,*

- (i) V_i is a subvariety of V_{i-1} ($i \geq 1$), where $V_0 = V$.
- (ii) V_i is also a smooth $(2, 3)$ -complete intersection of hypersurfaces in \mathbb{P}^{n+2-i} , $0 \leq i \leq n-1$,
- (iii) V_i has the property (\P_{n-i}) ,
- (iv) V_i has a D_6 -embedding.

The situation above is illustrated as follows:

$$\begin{array}{ccccccc}
 \mathbb{P}^{n+2} & \dashrightarrow & \mathbb{P}^{n+1} & \dashrightarrow & & \dashrightarrow & \mathbb{P}^4 & \dashrightarrow & \mathbb{P}^3 \\
 \cup & & \cup & & & & \cup & & \cup \\
 V & \supset & V_1 & \supset & \cdots & \supset & V_{n-2} & \supset & V_{n-1} \\
 \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 \mathbb{P}^n & \dashrightarrow & \mathbb{P}^{n-1} & \dashrightarrow & & \dashrightarrow & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1,
 \end{array}$$

where \dashrightarrow is a point projection, \downarrow is a triple covering, V_{n-2} and V_{n-1} are a $K3$ surface and a sextic curve, respectively.

Here we present examples.

Example 3.8. Let Δ be the smooth cubic in \mathbb{P}^3 defined by

$$F(X_0, X_1, X_2, X_3) = X_0^3 + X_1^3 + X_2^3 + X_0^2 X_3 + X_1 X_3^2 + X_3^3. \quad (7)$$

Let π_P be the projection from $P = (0 : 0 : 0 : 1)$ to the hyperplane \mathbb{P}^2 . Taking the affine coordinates $x = X_0/X_3$, $y = X_1/X_3$ and $z = X_2/X_3$, we get the defining equation of the affine part

$$f(x, y, z) = x^3 + y^3 + z^3 + x^2 + y + 1.$$

Put $x = at$, $y = bt$ and $z = ct$. Computing the discriminant $D(f)$ of $f(at, bt, ct) = (a^3 + b^3 + c^3)t^3 + a^2t^2 + bt + 1$ with respect to t , we obtain

$$D(f) = -(31a^6 - 18a^5b - a^4b^2 + 58a^3b^3 - 18a^2b^4 + 31b^6 + 54a^3c^3 - 18a^2bc^3 + 58b^3c^3 + 27c^6). \quad (8)$$

This yields the branch divisor of $\pi_P : \Delta \rightarrow \mathbb{P}^2$. From (7) and (8) we infer that the defining equation of $2R_1$ is

$$F(X_0, X_1, X_2, X_3) = 0 \text{ and } (X_0^2 + 2X_1X_3 + 3X_3^2)^2 = 0,$$

and that of R_2 is

$$F(X_0, X_1, X_2, X_3) = 0 \text{ and } 4X_0^2 - X_1^2 + 2X_1X_3 + 3X_3^2 = 0.$$

It is not difficult to check that R_2 is smooth and irreducible, hence P is a good point for Δ . By taking a double covering along this curve [8], we get the $K3$ surface Δ_P in \mathbb{P}^4 defined by $F = 0$ and $X_4^2 = 4X_0^2 - X_1^2 + 2X_1X_3 + 3X_3^2$, which is a $(2, 3)$ -complete intersection. The Galois line is given by $X_0 = X_1 = X_2 = 0$.

How is the Galois closure variety when the projection center is not a good point? Let us examine the following example.

Example 3.9. For a projection with some center $P \in \mathbb{P}^3 \setminus \Delta$, the Galois closure surface Δ_P is not necessarily a $K3$ surface. Indeed, let Δ be the smooth cubic defined by

$$F(X_0, X_1, X_2, X_3) = X_0^3 + X_1^3 + X_2^3 + X_0X_3^2 - X_3^3. \quad (9)$$

Clearly the point $P = (0 : 0 : 0 : 1)$ is not a Galois one. Taking the same affine coordinates as in Example 3.8, we get the defining equation of the affine part

$$f(x, y, z) = x^3 + y^3 + z^3 + x - 1.$$

Put $x = at$, $y = bt$ and $z = ct$. Computing the discriminant $D(f)$ of $f(at, bt, ct) = (a^3 + b^3 + c^3)t^3 + at - 1$ with respect to t , we obtain

$$D(f) = -(31a^3 + 27b^3 + 27c^3)(a^3 + b^3 + c^3). \quad (10)$$

This yields the branch divisor of $\pi_P : \Delta \rightarrow \mathbb{P}^2$. From (9) and (10) we infer that the defining equation of $2R_1$ is $C_1 + C_2$, where C_1 (resp. C_2) is given by $X_0^3 + X_1^3 + X_2^3 = 0$ (resp. $31X_0^3 + 27X_1^3 + 27X_2^3 = 0$). Hence the defining equation of the sextic $C_P(V)$ is

$$(X_0^3 + X_1^3 + X_2^3)(31X_0^3 + 27X_1^3 + 27X_2^3) = 0.$$

Let Δ_P be the double covering of Δ branched along the divisor R_2 , where $R_2 = R_{21} + R_{22}$ such that R_{21} (resp. R_{22}) is given by the intersection of $F = 0$ and $X_0 - X_3 = 0$ (resp. $F = 0$ and $X_0 - 3X_3 = 0$). The R_{2i} ($i = 1, 2$) is a smooth curve on Δ satisfying that $R_{2i}^2 = 3$, $R_2^2 = 12$ and $R_{21} \cdot R_{22} = 3$. We infer that Δ_P is a normal surface, therefore it is a Galois closure surface at P (Definition 1.2). However, it has three singular points of type A_1 , so that it is not a $K3$ surface. The minimal resolution of Δ_P turns out to be a $K3$ surface.

Remark 3.10. The variety with the property (\P_n) is not necessarily a $(2, 3)$ -complete intersection. For example, in case $n = 1$, Take $V = C$ as the Galois closure curve of a smooth cubic $\Delta \subset \mathbb{P}^2$ obtained as follows: let T be a tangent line to Δ at a flex. Choose a point $P \in T$ satisfying the following condition: if ℓ_P is a line passing through P and $\ell_P \neq T$, then ℓ_P does not tangent to Δ at any flex. Let C be the

Galois closure curve for the point projection $\pi_P : \Delta \rightarrow P^1$, i.e., $\tilde{\pi} : C \rightarrow \Delta$ is a double covering, which has four branch points (see, for example [5, pp. 287–288]), hence $g(C) = 3$. Let D be the divisor $\tilde{\pi}^*(\ell * \Delta)$, where ℓ is a line passing through P . Clearly we have $\deg D = 6$, the complete linear system $|D|$ has no base point and $\dim H^0(C, \mathcal{O}(D)) = 4$. Let $f : C \rightarrow C'$ be the morphism associated with $|D|$. The double covering $\tilde{\pi}$ factors as $\tilde{\pi} = \tilde{\pi}' \cdot f$, where $\tilde{\pi}' : C' \rightarrow \Delta$ is a restriction of the projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. Since $g(C') \geq 1$, we see $\deg C' \neq 2$ and 3 . Hence $\deg C' = 6$ and f is a birational morphism. Further, we have the projection $\tilde{\pi}' : C' \rightarrow \Delta$ and Δ is nonsingular, hence C' is smooth. Therefore f is an isomorphism. Since $g(C) = 3$, C is not a $(2, 3)$ -complete intersection.

4. PROOF

First we prove Theorem 3.1. The case $n = 1$ have been proved ([13]). So that we will restrict ourselves to the case $n \geq 2$.

Since V is embedded into \mathbb{P}^{n+2} associated with $|D|$, where D is a very ample divisor with $D^n = 6$, we can apply the results in Section 2. By assumption V has a Galois line ℓ such that the Galois group $G = G_\ell$ is isomorphic to D_6 . We can assume $G = \langle \sigma, \tau \rangle$ where $\sigma^3 = \tau^2 = 1$ and $\tau\sigma\tau = \sigma^{-1}$. Let $\rho_1 : V \rightarrow V^\tau = V/\langle \tau \rangle$. We see $\rho_2 : V^\tau \rightarrow V^\tau/G \cong \mathbb{P}^n$ turns out a morphism. Then, we have $\pi = \rho_2\rho_1 : V \rightarrow V/G \cong \mathbb{P}^n$. Note that ρ_2 is a non-Galois triple covering. By taking suitable coordinates, we can assume ℓ is given by $X_0 = X_1 = \cdots = X_n = 0$. As we see in Section 2, we have the representation $\beta : G \hookrightarrow PGL(n+3, k)$. Since the characteristic of k is zero, the projective representation is completely reducible, hence $\beta(\sigma)$ and $\beta(\tau)$ can be represented as

$$\mathbf{1}_{n+1} \oplus M_2(\sigma) \quad \text{and} \quad \mathbf{1}_{n+1} \oplus M_2(\tau),$$

respectively, where $M_2(\sigma)$ and $M_2(\tau)$ are in $GL(2, k)$. Since $G \cong D_6$, we have the representation

$$\beta(\sigma) = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ 0 & & & -\frac{1}{2} & \omega + \frac{1}{2} \\ & & & \omega + \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad \beta(\tau) = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & -1 \end{pmatrix},$$

where ω is a primitive cubic root of 1. Therefore, the fixed locus of τ is given by $f(V) \cap H$, where H is the hyperplane defined by $X_{n+2} = 0$. Put $Z = f(V) \cap H$, i.e., $Z \sim D$. Since Z is ample, it is connected. Looking at the representation $\beta(\tau)$, we see Z is smooth, hence it is a smooth irreducible variety. Take the point $P = (0 : \cdots : 0 : 1) \in \ell$ and an arbitrary point Q in V . Let ℓ_{PQ} be the line passing through P and Q . Then we have $\tau(\ell_{PQ}) = \ell_{PQ}$ and $\tau(V) = V$. Let π_P be the projection from the point P to the hyperplane H . Since Z is smooth, $\pi_P(V)$ is smooth. By Lemma 2.3, $\pi_P(V)$ is isomorphic to $V/\langle \tau \rangle$ and we may assume $\pi_P = \rho_1$. Therefore we see V is contained in the cone consisting of the lines passing through P and the points in V . Since $\deg V = 6$ and $\deg p = 2$, we conclude the variety V^τ is a smooth cubic in \mathbb{P}^{n+1} . This proves Theorem 3.1.

Next we prove Lemma 3.2. Let H_2 be a linear variety of dimension two and passing through P . If H_2 is general, then $\Delta \cap H_2$ is a smooth cubic in the plane $H_2 \cong \mathbb{P}^2$. Thus $C_P(\Delta) \cap H_2$ consists of six lines, hence we have $\deg C_P(\Delta) = 6$.

The proof of Proposition 3.4 is as follows. Suppose P is a general point for Δ and let π_P be the projection from P to the hyperplane \mathbb{P}^n . Put $B = \pi_P(R_2)$.

Claim 4.1. *The divisor B is irreducible.*

Proof. It is sufficient to check in a general affine part. Put $x_i = X_i/X_0$ ($i = 1, \dots, n+1$) and let $f(x_1, \dots, x_{n+1})$ be the defining equation of an affine part $X_0 \neq 0$ of Δ and $P = (u_1, \dots, u_{n+1}) \in \mathbb{A}^{n+1}$. Put

$$g(u_1, \dots, u_{n+1}, t_0, \dots, t_n, x) = f(u_1 + xt_0, \dots, u_{n+1} + xt_n),$$

where $(t_0, \dots, t_n) \in \mathbb{P}^n$. Let $D(g) = D(u_1, \dots, u_{n+1}, t_0, \dots, t_n)$ be the discriminant of g with respect to x . Owing to [9, Lemma 3] and [11, Claim 1], we see $D(g)$ is reduced and irreducible. Therefore for a general value $u_1 = a_1, \dots, u_{n+1} = a_{n+1}$, $D(a_1, \dots, a_{n+1}, t_0, \dots, t_n)$ is irreducible. This implies B is irreducible. \square

Claim 4.2. *The divisor R_2 is irreducible and smooth.*

Proof. Suppose R_2 is decomposed into irreducible components $R_{21} + \dots + R_{2r}$. Since B is irreducible, we have $\pi_P(R_{2i}) = B$ for each $1 \leq i \leq r$. However, since $\Delta * \ell$ has an expression $2P_1 + P_2$, the r must be 1. Thus R_2 is irreducible. Since $\Delta * \ell$ has an expression $2P_1 + P_2$, where $P_i \in \Delta$ ($i = 1, 2$), Δ and ℓ has a normal crossing at P_2 if $P_1 \neq P_2$. In case $P_1 = P_2$, the intersection number of Δ and ℓ at P_1 is three. Since $R_1 \ni P_1$ and $\Delta * C_P(\Delta) = 2R_1 + R_2$, we see that R_2 is smooth at P_1 . \square

This completes the proof of Proposition 3.4. The proof of Theorem 3.5 is as follows.

First note that P is not a Galois point. So we consider the Galois closure variety. The ramification divisor of $\pi_P : \Delta \rightarrow \mathbb{P}^n$ is $2R_1 + R_2$. The divisor R_2 is smooth and irreducible by assumption. Let Φ be the equation of the branch divisor of π_P . As we see in Example 3.8 ($a = X_0/tX_3$, $b = X_1/tX_3$, $c = X_2/tX_3$) the discriminant is given by the homogeneous equation of X_0, \dots, X_n , hence we infer that $\pi_P^*(\Phi)$ has the expression as $\Phi_1^2 \cdot \Phi_2$, where $\Phi_2 = 0$ defines R_2 . Since $\deg \Phi_2 = 2$, we can define the variety in \mathbb{P}^{n+2} by $F = 0$ and $X_{n+2}^2 = \Phi_2$, which is smooth and turns out to be the Galois closure variety. This proves Theorem 3.5.

We go to the proof of Proposition 3.7. Let H be a general hyperplane containing the Galois line ℓ for V in Theorem 3.1. Put $V_1 = V \cap H$ and $D_1 = D \cap H$. Since we are assuming $n \geq 2$, the V_1 is irreducible and nonsingular by Bertini's theorem. Thus, we have $\dim V_1 = n - 1$, $D_1^{n-1} = 6$ and V_1 is also a smooth $(2, 3)$ -complete intersection. Note that $V_1 \sim D$ on V . Thus we have the exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_V(V_1) \rightarrow \mathcal{O}_{V_1}(D_1) \rightarrow 0.$$

Taking cohomology, we get a long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(V, \mathcal{O}_V) &\rightarrow H^0(V, \mathcal{O}_V(V_1)) \rightarrow H^0(V_1, \mathcal{O}_{V_1}(D_1)) \\ &\rightarrow H^1(V, \mathcal{O}_V) \rightarrow \dots \end{aligned}$$

Since V is the complete intersection, we have $H^1(V, \mathcal{O}_V) = 0$ (cf. [2, III, Ex. 5.5]). Then V_1 has the same properties as V does, i.e., $\dim V_1 = n - 1$, $D_1^{n-1} = 6$,

$\dim H^0(V_1, \mathcal{O}(D_1)) = n + 2$ and ℓ is a Galois line for V_1 and the Galois group is isomorphic to D_6 . Continuing the Descending Procedure, we get the sequence of Proposition 3.7.

There are a lot of problems concerning our theme, we pick up some of them.

Problems.

- (1) For each finite subgroup G of $GL(2, k)$, does there exist a pair (V, D) which defines the Galois embedding with the Galois group G such that $D^n = |G|$, $\dim V = n$ and $\dim H^0(V, \mathcal{O}(D)) = n + 3$?
- (2) How many Galois subspaces do there exist for one Galois embedding? In case a smooth hypersurface V in \mathbb{P}^{n+1} , there exist at most $n + 2$. Further, it is $n + 2$ if and only if V is Fermat variety [11].
- (3) Does there exist a variety V on which there exist two divisors D_i ($i = 1, 2$) such that they give Galois embeddings and $D_1^n \neq D_2^n$?

For the detail, please visit our website

<http://mathweb.sc.niigata-u.ac.jp/~yoshihara/openquestion.html>

REFERENCES

1. J. Harris, Galois groups of enumerative problems, *Duke Math. J.*, **46** (1979), 685–724.
2. R. Hartshorne, Algebraic Geometry, *Graduate Texts in Mathematics*, **52** Springer-Verlag.
3. S. Iitaka, Algebraic Geometry, An introduction to birational geometry of algebraic varieties, *Graduate Texts in Mathematics*, **76** Springer-Verlag.
4. G. Malle and B. H. Matzat, Inverse Galois Theory, *Springer Monogr., Math.*, Springer-Verlag, New York, Heidelberg, Berlin, 1999.
5. K. Miura and H. Yoshihara, Field theory for function fields of plane quartic curves, *J. Algebra*, **226** (2000), 283–294.
6. M. Namba, Branched coverings and algebraic functions, *Pitman Research Notes in Mathematics*, Series 161.
7. R. Shafarevich, Basic Algebraic Geometry I, Second, Revised and Expanded Edition, Springer-Verlag.
8. H. Tokunaga, Triple coverings of algebraic surfaces according to the Cardano formula, *J. Math. Kyoto University*, **31** (1991), 359–375.
9. H. Yoshihara, Function field theory of plane curves by dual curves, *J. Algebra*, **239** (2001), 340–355.
10. ———, Galois points on quartic surfaces, *J. Math. Soc. Japan*, **53** (2001), 731–743.
11. ———, Galois points for smooth hypersurfaces, *J. Algebra*, **264** (2003), 520–534.
12. ———, Families of Galois closure curves for plane quartic curves, *J. Math. Kyoto Univ.*, **43** (2003), 651–659.
13. ———, Galois lines for space curves, *Algebra Colloquium*, **13** (2006), 455–469.
14. ———, Galois embedding of algebraic variety and its application to abelian surface, *Rend. Sem. Mat. Univ. Padova*, **117** (2007), 69–86.
15. ———, Galois lines for normal elliptic space curves, II, *Algebra Colloquium* **19**, 867–876 (2012).